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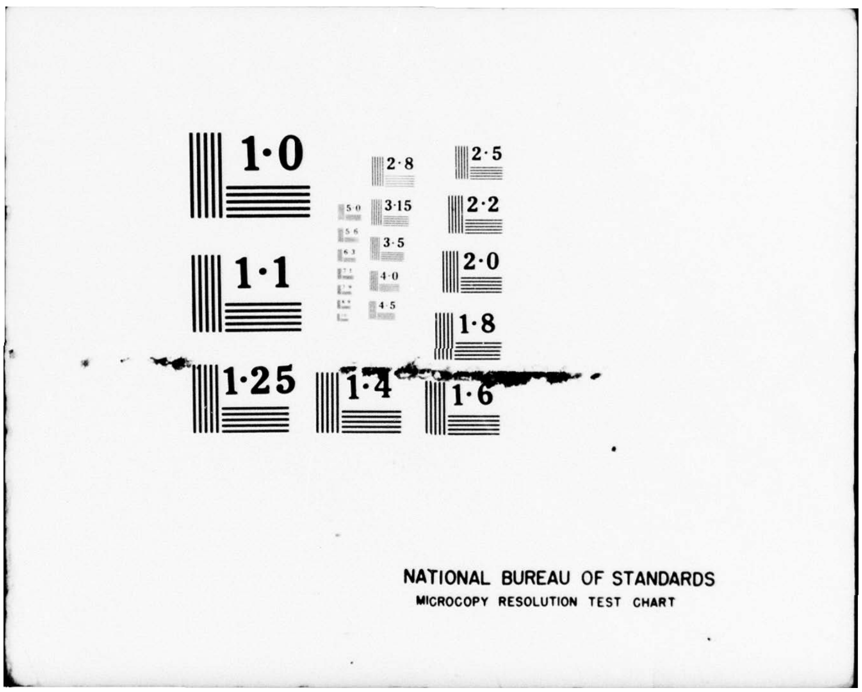
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Technical Report 162

THE ALE OUTPUT COVARIANCE FUNCTION FOR A SINUSOID IN UNCORRELATED NOISE

ST Alexander

1 August 1977

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Naval Electronic Systems Command

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AN ACTIVITY OF THE NAVAL MATERIAL COMMAND

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The output covariance function of the Adaptive Line Enhancer (ALE) is derived and discussed in detail. The function of the linear prediction filter in the ALE structure is illustrated and the expression for the converged filter coefficients is seen to be accurately represented by the Wiener solution. Properties of the covariance function are discussed in relation to signal frequency and filter length and the filter output is shown to contain a small-magnitude nonstationary component. However, the component has negligible effect →

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in cases of interest.

The use of the ALE as a detection prefilter is summarized. When the covariance function is known it is possible to determine the detection performance of the ALE if detection statistic means and variances are known. The specific example of deriving ALE output DFT means and variances bases on knowledge of the covariance function is then presented.

CONTENTS

1. INTRODUCTION

This paper derives the output covariance function of the Adaptive Line Enhancer (ALE) for the case of a single known sinusoid in uncorrelated noise. An important application of this function arises when deriving the detection performance of a processor incorporating the ALE as a prefilter. For this case, it is necessary to derive expressions for the means and variances of the spectral estimates (DFT bin components) at the output of the processor. When the covariance function of the ALE output is known, it is possible to compute these spectral means and variances.

This paper contains an expansion of the derivations by Medaugh[1] for the covariance function and does not assume a necessarily long filter length compared to signal period nor that the signal frequency be bin-centered. In addition, the mean and variance derivation contained herein are extended to the case $K \leq L$ where L is the adaptive filter length and K is the number of DFT points. The expressions for covariance function and filter output DFT means and variances as derived in reference 1 are shown to be correct as long as the following conditions are met:

1. The filter length L equals the number of points K in the DFT,

2. The signal frequency ω_0 is given by $\omega_0 = \pi n/L$ (n = integer), where L is the filter length and

3. The signal frequency ω_0 is not allowed to approach closely $\omega = 0$ or $\omega = \pi$ radians/sampling interval. This corresponds physically to ω_0 bounded by

$$\frac{\pi}{L} < \omega_0 < \pi\left(\frac{L-1}{L}\right).$$

A general expression for the output covariance function is developed when these conditions are not met. An interesting result occurs when condition 2 is not met; the filter output contains a small-magnitude nonstationary component due to the finite length of the filter.

Section 2 provides a brief introduction to the ALE and outlines its operational characteristics. A specific application of the ALE is presented and a motivation is given to obtain the output covariance function. Section 3 provides a detailed derivation and discussion of the covariance function and gives examples of limiting cases and first-order approximations. Section 4 applies the covariance function to a specific application to find the spectral means and variances of ALE output DFTs.

2. THE ADAPTIVE LINE ENHANCER (ALE)

PROPERTIES OF THE ALE

The Adaptive Line Enhancer (ALE) is a processor designed to separate the relatively narrowband signals (i.e., sinusoids) contained in an input sequence from the relatively broadband components (i.e., broadband noise) in the input. A block diagram of the ALE, shown in Figure 1, clearly defines the implementation of a linear prediction filter in the lower channel. The operation of the ALE has been discussed in detail in the references cited here (2, 3, 4, 5) and only will be summarized at this time. The basic principle of operation is: the narrowband components of the input sequence are strongly correlated over time and thus may be predicted with accuracy by the linear prediction filter. The broadband components of the input, however, have appreciable correlation only over a short time and are not predictable with any degree of accuracy. The input sequence is split into two channels: one is delayed by Δ sampling periods (denoted by the $Z^{-\Delta}$ block) and then input to the linear prediction filter; the other channel is fed directly to a summing junction from which the filter output is subtracted. The linear prediction filter consists of an L tap tapped-delay-line (TDL), the taps of which are multiplied by the L weights of the adaptive weight vector (see Figure 2).

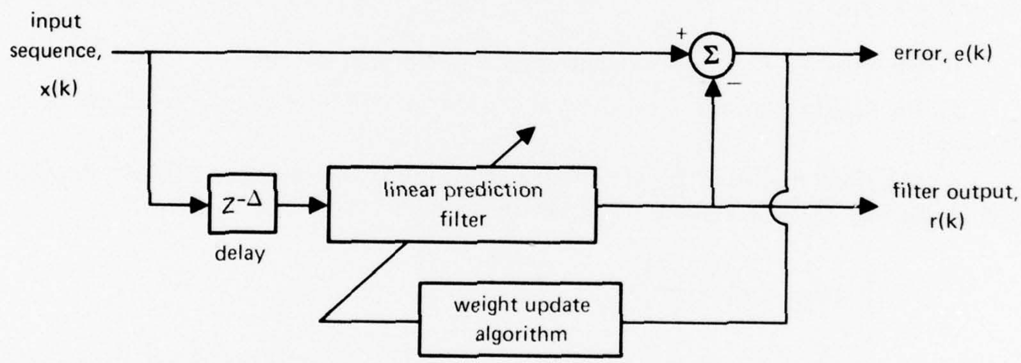

Figure 1. Block diagram of the Adaptive Line Enhancer (ALE).

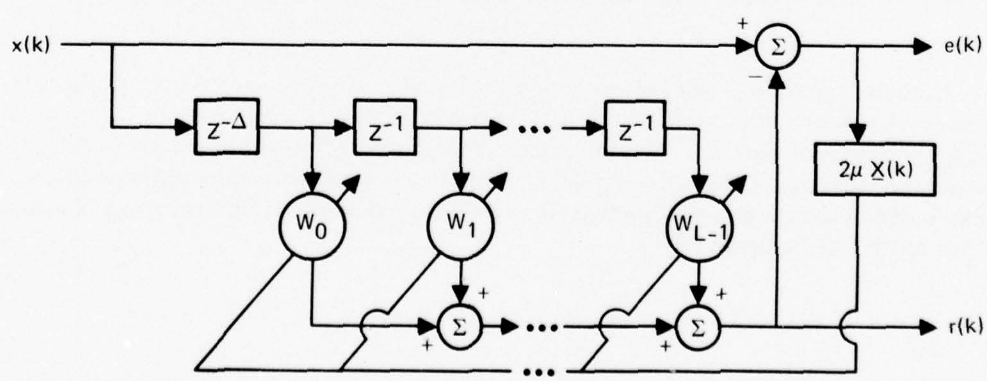

Figure 2. Expanded block diagram of ALE.

The weights are denoted by the W_j multipliers and these products are then summed to produce the filter output at time k, denoted by r(k). Mathematically, the filter output is given by the convolution sum

$$r(k) = \sum_{j=0}^{L-1} W_j(k)\, x(k - j - \Delta) \tag{1}$$

where the filter weights have been written as a function of time argument k to reflect their adaptive nature. The filter output is then fed into a summing junction and subtracted from the original data sequence. The purpose of the filter in the delayed channel is to predict the current value of the data input, x(k), based upon a linear combination of past input samples $x(k - \Delta)$ through $x(k - \Delta - L + 1)$. The filter is described as linear prediction, since the filter output r(k) is a prediction, $\hat{x}(k)$, of the current actual input sample x(k). Thus the output of the summing junction, denoted by e(k), may be thought of as the error between the actual value x(k) and estimate $\hat{x}(k)$:

$$e(k) = x(k) - \hat{x}(k). \tag{2}$$

If the criteria for the prediction filter is to minimize the mean square of this error, then it is well known[6] that the resultant filter weights $\underline{W}^*$ (or coefficients) must satisfy the discrete Wiener-Hopf matrix equation:

$$\underline{W}^* = R^{-1}\, \underline{P} \tag{3}$$

where R is the L × L autocorrelation matrix of the input sequence and $\underline{P}$ is the L × 1 autocorrelation vector with autocorrelation elements

$$\underline{P} = \begin{bmatrix} r_{xx}(\Delta) \\ r_{xx}(\Delta+1) \\ \cdot \\ \cdot \\ \cdot \\ r_{xx}(\Delta + L - 1) \end{bmatrix} \tag{4}$$

To solve equation 3 for the $\underline{W}^*$, sometimes called the Wiener solution or optimal weight vector, requires perfect knowledge of the autocorrelation lags $r_{xx}(\ell)$, but generally these are never known exactly. The ALE obviates this requirement by solving for the $\underline{W}^*$ via an iterative technique called the LMS (for least mean square) algorithm. This is a gradient descent type of algorithm which has been shown[7, 8] to converge to the Wiener solution for uncorrelated stationary inputs. The algorithm starts with an arbitrary initial weight vector $\underline{W}(0)$ and updates the weight vector elements through the recursion

$$\underline{W}(k+1) = \underline{W}(k) + 2\mu e(k)\, \underline{X}(k). \tag{5}$$

In equation 5, $\underline{X}(k)$ is the vector consisting of the contents of the TDL at time k, e(k) is the error signal at time k and μ is a feedback constant which controls the speed of adaptation. A stability requirement on μ (references 7, 8) is that it be bounded by

$$o < \mu < 1/\lambda_{max} \tag{6}$$

where λ_{max} is the largest eigenvalue of the input autocorrelation matrix R.

Figure 2 shows that the error value e(k) is multiplied by the vector $2\mu\underline{X}(k)$ and then used to update the individual W_j according to the LMS algorithm. During the adaptation period, the algorithm updates the filter coefficients such that the filter output converges toward the highly correlated (and thus highly predictable) components of the input. The minimization of mean-square-error criteria aligns the phase of the narrowband components in the data sequence and filter output, which, after convergence, removes the narrowband components from the error sequence e(k). This is the steady-state operation of the filter in which the filter weights may be represented by the Wiener solution, as shown analytically[2] and experimentally.[3]

In steady-state, the weight vector converges in the mean to the Wiener solution. However, at any one given time k the instantaneous weight value $W_j(k)$ is given by

$$W_j(k) = W_j^* + V_j(k) \tag{7}$$

where W_j^* is the j^{th} element of the optimal weight vector and $V_j(k)$ is a zero-mean gaussian noise component with variance given by μc^2 (c^2 = total input power, signal plus noise). This component is called the misadjustment noise and is due to the use of the LMS algorithm as a finite length approximation to the true gradient (reference 2). Furthermore, reference 2 shows the misadjustment noise is uncorrelated from weight-to-weight; that is,

$$E[V_i(k)\, V_j(k)] = \mu c^2\, \delta(i-j), \text{ for all i, j.} \tag{8}$$

The misadjustment noise and the converged weights W_i^* are also uncorrelated. Therefore, due to the zero-mean property of the $V_j(k)$, and that the W_i^* are deterministic:

$$E[W_i^*\, V_j(k)] = W_i^*\, E[V_j(k)] = 0; \text{ for all i, j.} \tag{9}$$

Since the variance of $V_j(k)$ is proportional to μ, it is evident that the variance of the misadjustment noise can be made arbitrarily small by decreasing μ. This translates into an ability to approximate the Wiener solution $\underline{W}^*$ to within any desired bound, with the trade-off that the LMS algorithm takes increasingly longer to converge. For many problems of practical interest, however, this is of no consequence, and thus this paper will treat the converged weight vector as the Wiener solution $\underline{W}^*$.

AN APPLICATION OF THE ALE AS A DETECTION PREFILTER

The ALE provides a real-time method of separating the narrowband input signals from the corruptive broadband noise components. A primary function of the ALE is as a detection prefilter as shown in Figure 3. In this implementation, the narrowband components

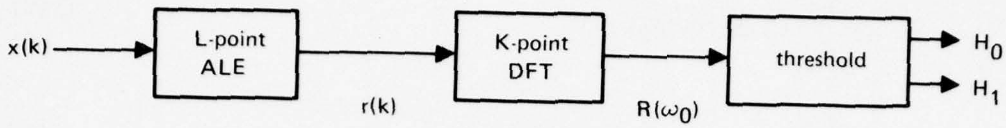

Figure 3. Use of the ALE as a predetection filter.

appear at the filter output r(k), which is then used as the input to a DFT. Detection of a signal with frequency ω_0 is performed by examining the magnitude square of the ω_0 frequency component and comparing it to a predetermined threshold level. Introductions to the detection problem in general may be found in many texts[9, 10] and only the framework of the problem is presented here to provide motivation for obtaining the covariance function.

The case examined here is a purely sinusoidal signal s(k) corrupted by an additive gaussian noise sequence, n(k). Let s(k) be defined by

$$s(k) = A \sin(\omega_0 k + \theta) \tag{10}$$

where A is the signal amplitude, ω_0 is the signal frequency and θ is the unknown initial phase of the sinusoid. Let the additive noise n(k) be a white zero-mean sequence with variance ν^2; that is

$$E[n(k)] = 0 \tag{11a}$$

$$E[n(k)\,n(p)] = \nu^2\,\delta(k-p) \tag{11b}$$

where $E[\cdot]$ denotes the expectation operator and $\delta(\cdot)$ is the Kronecker delta operator. With the signal term and noise sequence represented by the preceding equations, the received data sequence x(k) becomes

$$x(k) = s(k) + n(k). \tag{12}$$

Now consider the operation of the converged prediction filter upon x(k). Since s(k) is completely deterministic and zero-mean, the variable x(k) is a zero-mean gaussian random variable for all values of k. The filter output, denoted by $r(k|\theta)$ to reflect the dependence on unknown initial phase θ, is defined as

$$r(k|\theta) = \sum_{j=0}^{L-1} W_j^*\, x(k-j-\Delta). \tag{13}$$

Thus $r(k|\theta)$ is a linear combination of previous input samples. The summation of the L gaussian variables, therefore, produces the gaussian variable $r(k|\theta)$. As shown in Figure 3, the detection processor then operates upon the filter output with a K-point DFT to produce the spectral estimate $R(\omega)$. The ω_0 component of this estimate, $R(\omega_0)$, is given by

$$R(\omega_0) = u + jv \tag{14}$$

where

$$u = \sum_{k=0}^{K-1} r(k|\theta) \cos \omega_0 k \tag{15a}$$

$$v = \sum_{k=0}^{K-1} r(k|\theta) \sin \omega_0 k. \tag{15b}$$

The variables u and v as defined in equations 14 and 15 are recognized to be the real and imaginary components of the spectral estimate at frequency ω_0. The statistical independence of u and v has been demonstrated in reference 1, and thus have a joint probability density p(u, v) given by

$$p(u, v) = p(u)\, p(v). \tag{16}$$

It may be seen from equations 15a and 15b that for a given ω_0 the variables u and v are formed by linear combinations of the output samples. Since the individual $r(k|\theta)$ are gaussian variables, the scaling by cos $\omega_0 k$ and summation cause u to be a gaussian variable. A similar argument holds for v. Thus the u, v are gaussian random variables with calculable means $\bar{u}$ and $\bar{v}$ and variances σ_u^2 and σ_v^2, which completely specifies the joint probability density of equation 16:

$$p(u, v) = \frac{1}{\sqrt{2\pi}\sigma_u} \exp\left[-\frac{1}{2}\left(\frac{u - \bar{u}}{\sigma_u}\right)^2\right] \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left[-\frac{1}{2}\left(\frac{v - \bar{v}}{\sigma_v}\right)^2\right] \tag{17}$$

$$p(u, v) = \frac{1}{2\pi\sigma_u\sigma_v} \exp\left[-\frac{1}{2}\left(\frac{u - \bar{u}}{\sigma_u}\right)^2 - \frac{1}{2}\left(\frac{v - \bar{v}}{\sigma_v}\right)^2\right]. \tag{18}$$

Thus, if one has knowledge of the mean values $\bar{u}$ and $\bar{v}$ (defined by E(u) and E(v), respectively) and the variances σ_u^2 and σ_v^2, the joint probability density function is immediately known.

Consider first the variable u. The variance of u is defined by σ_u^2, where

$$\sigma_u^2 = E\{u^2\} - [E\{u\}]^2. \tag{19}$$

By applying the definition in equation 15a to equation 19, σ_u^2 becomes

$$\sigma_u^2 = E\left\{\sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} r(k|\theta)\, r(\ell|\theta) \cos \omega_0 k \cos \omega_0 \ell\right\}$$

$$- E\left\{\sum_{k=0}^{K-1} r(k|\theta) \cos \omega_0 k\right\} E\left\{\sum_{\ell=0}^{K-1} r(\ell|\theta) \cos \omega_0 \ell\right\}$$

$$= \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} E\{r(k|\theta)\, r(\ell|\theta)\} \cos \omega_0 k \cos \omega_0 \ell$$

$$- \sum_{k=0}^{K-1} E\{r(k|\theta)\} \cos \omega_0 k \sum_{\ell=0}^{K-1} E\{r(\ell|\theta)\} \cos \omega_0 l$$

or

$$\sigma_u^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} \left[E\{r(k|\theta)\, r(\ell|\theta)\} - E\{r(k|\theta)\}\, E\{r(\ell|\theta)\} \right] \cos \omega_0 k \cos \omega_0 \ell. \tag{20}$$

The autocorrelation function, $\phi_r(m)$ is defined as

$$\phi_r(m) = E\{r(k|\theta)\, r(k + m|\theta)\} \tag{21}$$

and the covariance function, $\gamma_r(m)$, as

$$\gamma_r(m) = \phi_r(m) - E\{r(k|\theta)\}\, E\{r(k + m|\theta)\}. \tag{22}$$

Then, the expression for the variance of the statistic u becomes

$$\sigma_u^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} \gamma_r(\ell - k) \cos \omega_0 k \cos \omega_0 \ell \tag{23}$$

Similarly, one may derive for the statistic v the following expression:

$$\sigma_v^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} \gamma_r(\ell - k) \sin \omega_0 k \sin \omega_0 \ell \tag{24}$$

Thus the variances of the u and v statistics are dependent upon finding the covariance function $\gamma_r(\ell - k)$. This derivation is pursued in the following section.

3. DERIVATION OF THE ALE OUTPUT COVARIANCE FUNCTION

First, the expression is found for the expectation of the output at time k conditioned on a knowledge of θ, denoted by $E\{r(k|\theta)\}$:

$$E\{r(k|\theta)\} = E\left\{\sum_{j=0}^{L-1} W_j(k)\, x(k-j-\Delta)\right\}, \qquad 0 \leqslant j \leqslant L-1$$

$$= E\left\{\sum_{j=0}^{L-1} [W_j^* + V_j(k)]\, [A \sin[\omega_0(k-j-\Delta)+\theta] + n(k-j-\Delta)]\right\}. \tag{25}$$

However, the misadjustment noise $V_j(k)$ and white noise sequence n(p) are uncorrelated; that is,

$$E\{V_j(k)\, n(p)\} = E\{V_j(k)\}\, E\{n(p)\} = 0$$

for all k, j and p. Furthermore, since W_j^* is deterministic,

$$E\{W_j^*\, n(p)\} = W_j^*\, E\{n(p)\} = 0.$$

Similarly, the sinusoidal term is completely deterministic, causing the cross product expectation between $V_j(k)$ and the sinusoid to vanish. This reduces equation 25 to the following:

$$E\{r(k|\theta)\} = \sum_{j=0}^{L-1} W_j^*\, A \sin[\omega_0(k-j-\Delta)+\theta]. \tag{26a}$$

A similar derivation gives the expectation $E\{r(k+m|\theta)\}$ as

$$E\{r(k+m|\theta)\} = \sum_{i=0}^{L-1} W_i^*\, A \sin[\omega_0(k+m-i-\Delta)+\theta]. \tag{26b}$$

Using equation 22 as the definition of the covariance function, then the evaluation of $\phi_r(m)$, as given by equation 21 is:

$$\phi_r(m) = E\{r(k|\theta)\, r(k+m|\theta)\}, \qquad L = \text{filter length},$$

$$= E\left\{\sum_{j=0}^{L-1} W_j(k)\, x(k-j-\Delta) \sum_{i=0}^{L-1} W_i(k+m)\, x(k+m-i-\Delta)\right\}$$

$$\phi_r(m) = E\left\{\sum_{j=0}^{L-1} \sum_{i=0}^{L-1} [W_j(k)\, W_i(k+m)]\ [A \sin [\omega_0(k-j-\Delta)+\theta] + n(k-j-\Delta)] \times [A \sin [\omega_0(k+m-i-\Delta)+\theta] + n(k+m-i-\Delta)]\right\} \tag{27}$$

As before, the misadjustment error, $V_j(k)$, the converged weight value W_j^*, and the white noise sequence $n(k)$ are all uncorrelated. This causes several of the cross-products in equation 27 to vanish, which will simplify the expression. Expanding equation 27:

$$\phi_r(m) = \phi_1 + \phi_2 + \phi_3 + \phi_4 \tag{28}$$

where

$$\phi_1 = E\left\{\sum_{j=0}^{L-1} \sum_{i=0}^{L-1} [W_j(k)\, W_i(k+m)]\ A^2 \sin [\omega_0(k-j-\Delta)+\theta] \sin [\omega_0(k+m-i-\Delta)+\theta]\right\} \tag{29}$$

$$\phi_2 = E\left\{\sum_{j=0}^{L-1} \sum_{i=0}^{L-1} [W_j(k)\, W_i(k+m)]\ A \sin [\omega_0(k-j-\Delta)+\theta]\ n(k+m-i-\Delta)\right\} \tag{30}$$

$$\phi_3 = E\left\{\sum_{j=0}^{L-1} \sum_{i=0}^{L-1} [W_j(k)\, W_i(k+m)]\ A \sin [\omega_0(k+m-i-\Delta)+\theta]\ n(k-j-\Delta)\right\} \tag{31}$$

$$\phi_4 = E\left\{\sum_{j=0}^{L-1} \sum_{i=0}^{L-1} [W_j(k)\, W_i(k+m)]\ n(k-j-\Delta)\, n(k+m-i-\Delta)\right\}. \tag{32}$$

First, evaluate ϕ_2:

$$\phi_2 = E\left\{\sum_{j=0}^{L-1} \sum_{i=0}^{L-1} [W_j(k)\, W_i(k+m)]\ A \sin [\omega_0(k-j-\Delta)+\theta]\ n(k+m-i-\Delta)\right\}$$

$$= A \sum_{j=0}^{L-1} \sum_{i=0}^{L-1} E\left\{[W_j^* + V_j(k)]\ [W_i^* + V_i(k+m)]\ n(k+m-i-\Delta)\right\} \times \sin [\omega_0(k-j-\Delta)+\theta]$$

$$= A \sum_{j=0}^{L-1} \sum_{i=0}^{L-1} E\Big\{[W_j{}^* W_i{}^* + V_j(k)\, W_i{}^* + W_j{}^* V_i(k+m) + V_j(k)\, V_i(k+m)]$$

$$\times\, n(k+m-i-\Delta)\Big\} \times \sin\,[\omega_0(k-j-\Delta)+\theta]. \qquad (33)$$

However, it has been assumed that $W_j{}^*$, $V_j(k)$ and $n(k)$ are all uncorrelated for all j and k. This causes the expectation in equation 33 to vanish giving

$$\phi_2 = 0.$$

Similarly, it may be calculated that $\phi_3 = 0$.

Now evaluate ϕ_1 in equation 29:

$$\phi_1 = E\left\{\sum_{j=0}^{L-1} \sum_{i=0}^{L-1} [W_j(k)\, W_i(k+m)]\ A^2 \sin\,[\omega_0(k-j-\Delta)+\theta]\right.$$

$$\left. \times \sin\,[\omega_0(k+m-i-\Delta)+\theta]\right\}$$

$$= A^2 \sum_{j=0}^{L-1} \sum_{i=0}^{L-1} E\Big\{[W_j{}^* + V_j(k)]\ [W_j{}^* + V_i(k+m)]\Big\}$$

$$\times \sin\,[\omega_0(k-j-\Delta)+\theta]\ \sin\,[\omega_0(k+m-i-\Delta)+\theta].$$

Again $W_j{}^*$ and $V_j(k)$ are uncorrelated and $E\{W_j{}^* V_j(k)\} = 0$ giving

$$\phi_1 = A^2 \sum_{j=0}^{L-1} \sum_{i=0}^{L-1} E[W_j{}^* W_i{}^* + V_j(k)\, V_i(k+m)]\ \sin\,[\omega_0(k-j-\Delta)+\theta]$$

$$\times \sin\,[\omega_0(k+m-i-\Delta)+\theta]. \qquad (34)$$

For the sinusoid in white noise, it has been shown (references 3 and 4) that the converged weights $W_j{}^*$ have the form

$$W_j{}^* = \frac{2a^*}{L} \cos \omega_0(j+\Delta) \qquad (35a)$$

where

$$a^* = \frac{(L/2)\ \mathrm{SNR}}{1 + (L/2)\ \mathrm{SNR}} \qquad (35b)$$

and

$$\mathrm{SNR} = A^2/2\nu^2.$$

Thus, we have $E[W_j^* W_i^*] = W_j^* W_i^*$. Furthermore, the misadjustment noise is assumed to be uncorrelated from weight-to-weight but to have a constant variance over time:

$$E[V_j(k)\, V_i(k+m)] = \mu c^2\, \delta(i-j). \tag{36}$$

Substituting equation 36 into equation 34 and rearranging, then:

$$\begin{aligned}
\phi_1 &= \sum_{j=0}^{L-1}\sum_{i=0}^{L-1} A^2 W_i^* W_j^* \sin[\omega_0(k-j-\Delta)+\theta]\sin[\omega_0(k+m-i-\Delta)+\theta] \\
&\quad + A^2 \sum_{j=0}^{L-1}\sum_{i=0}^{L-1} \mu c^2\, \delta(i-j)\sin[\omega_0(k-j-\Delta)+\theta] \\
&\quad \times \sin[\omega_0(k+m-i-\Delta)+\theta] \\
&= \sum_{j=0}^{L-1} W_j^* A \sin[\omega_0(k-j-\Delta)+\theta] \sum_{i=0}^{L-1} W_i^* A \\
&\quad \times \sin[\omega_0(k+m-i-\Delta)+\theta] \\
&\quad + \frac{A^2}{2}\sum_{j=0}^{L-1}\sum_{i=0}^{L-1} \mu c^2\, \delta(i-j)\Big[\cos\omega_0(m-i+j) \\
&\quad - \cos[\omega_0(2k+m-i-j-2\Delta)+2\theta]\Big].
\end{aligned} \tag{37}$$

But comparing this expression to equations 25 and 26 it is seen that

$$\begin{aligned}
\phi_1 &= E\{r(k|\theta)\}\; E\{r(k+m|\theta)\} \\
&\quad + \frac{A^2\mu c^2}{2}\sum_{j=0}^{L-1}\sum_{i=0}^{L-1} \delta(i-j)\Big\{\cos\omega_0(m-i+j) \\
&\quad - \cos[\omega_0(2k+m-i-j-2\Delta)+2\theta]\Big\} \\
&= E\{r(k|\theta)\}\; E\{r(k+m|\theta)\} \\
&\quad + \frac{A^2\mu c^2}{2}\sum_{j=0}^{L-1}\Big\{\cos\omega_0 m - \cos[\omega_0(2j-2k+m+2\Delta)-2\theta]\Big\}
\end{aligned}$$

$$= E\left\{r(k|\theta)\right\} E\left\{r(k+m|\theta)\right\} + \frac{A^2 \mu c^2 L}{2} \cos \omega_0 m$$

$$- \frac{A^2 \mu c^2}{2} \sum_{j=0}^{L-1} \cos\left[2\omega_0 j + (m - 2k + 2\Delta)\,\omega_0 - 2\theta\right]. \qquad (38)$$

Formulas are available to evaluate a finite sum[11] and several are listed in the appendix. The last term in equation 38 may be written in the form

$$\sum_{j=0}^{L-1} \cos\left[2\omega_0 j + \psi\right] = \frac{\sin \omega_0 L}{\sin \omega_0} \cos\left[\omega_0 (L-1) + \psi\right]$$

where $\psi = (m - 2k + 2\Delta)\,\omega_0 - 2\theta$. Thus equation 38 becomes

$$\phi_1 = E\left\{r(k|\theta)\right\} E\left\{r(k+m|\theta)\right\} + \frac{A^2 \mu c^2 L}{2} \cos \omega_0 m$$

$$- \frac{A^2 \mu c^2}{2} S_L(\omega_0) \cos\left[\omega_0 (L-1) + (m - 2k + 2\Delta)\,\omega_0 - 2\theta\right] \qquad (39)$$

where the substitution

$$S_L(\omega_0) = \frac{\sin \omega_0 L}{\sin \omega_0}$$

has been made. For n(k) a white, gaussian noise sequence, $\Delta = 1$ is sufficient to decorrelate signal and noise components. Thus equation 39 simplifies to

$$\phi_1 = E\left\{r(k|\theta)\right\} E\left\{r(k+m|\theta)\right\} + \frac{A^2 \mu c^2 L}{2} \cos \omega_0 m$$

$$- \frac{A^2 \mu c^2}{2} S_L(\omega_0) \cos\left[\omega_0 (2k - m - L - 1) + 2\theta\right]. \qquad (40)$$

This differs from the expression obtained by Medaugh in reference 1 by the inclusion of the final term in equation 40. Implications of this will be examined later.

Returning to equation 28, it remains to evaluate ϕ_4, given by

$$\phi_4 = E\left\{\sum_{j=0}^{L-1} \sum_{i=0}^{L-1} \left[W_j(k)\, W_i(k+m)\right] n(k - j - \Delta)\, n(k + m - i - \Delta)\right\}.$$

Thus,

$$\phi_4 = \sum_{j=0}^{L-1} \sum_{i=0}^{L-1} E\left\{W_j(k)\, W_i(k+m)\right\} E\left\{n(k - j - \Delta)\, n(k + m - i - \Delta)\right\}$$

$$= \sum_{j=0}^{L-1} \sum_{i=0}^{L-1} [W_j{}^* W_i{}^* + E\{V_j(k)\, V_i(k+m)\}]\; E\{n(k-j-\Delta)$$

$$\times\, n(k+m-i-\Delta)\}.$$

The noise sequence n(k) is white, zero-mean and gaussian with variance ν^2. Thus,

$$\phi_4 = \sum_{j=0}^{L-1} \sum_{i=0}^{L-1} [W_j{}^* W_i{}^* \nu^2 \delta(m-i+j) + \mu c^2 \delta(i-j)\, \nu^2 \delta(m-i+j)]$$

$$= \sum_{j=0}^{L-1} \sum_{i=0}^{L-1} \frac{4a^{*2}\nu^2}{L^2} \cos[\omega_0(j+\Delta)] \cos[\omega_0(i+\Delta)]\; \delta(m-i+j)$$

$$+ \sum_{j=0}^{L-1} \mu c^2 \nu^2 \delta(m)$$

$$\phi_4 = \frac{2a^{*2}\nu^2}{L^2} \sum_{j=0}^{L-1} \sum_{i=0}^{L-1} [\cos \omega_0(i-j) + \cos \omega_0(i+j+2\Delta)]\; \delta(m-i+j)$$

$$+ \mu c^2 \nu^2 L\, \delta(m). \tag{41}$$

Now sum over the i-index in the first term of equation 41. The impulse function is non-zero only when i = m + j, and the double-summation reduces to

$$\sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \{\cos \omega_0(i-j) + \cos \omega_0(i+j+2\Delta)\}\; \delta(m-i+j)$$

$$= \sum_{i} \{\cos \omega_0(m) + \cos \omega_0(m+2i+2\Delta)\}, \tag{42}$$

$$0 \leqslant |m| \leqslant L-1.$$

The index on i no longer goes from 0 to L − 1 since not every value of $j = |m| + i$ lies in the interval $0 \leqslant |m| + i \leqslant L - 1$. There are only $j - |m|$ values of i for which the delta function in equation 42 will be non-zero. The index i thus has the value $i = j - |m|$, and

when $j = 0 \rightarrow i = -|m|$,

when $j = L - 1 \rightarrow i = L - 1 - |m|$.

Furthermore, the lower bound on i is limited to i = 0 and the upper bound is i = L - 1. Since m can be positive or negative lag in the range $|m| \leq L - 1$, the summation in equation 42 may be written as either:

$$I = \sum_{i=0}^{L-1-m} \left\{\cos \omega_0 m + \cos \omega_0 (m + 2i + 2\Delta)\right\}, \qquad 0 \leq m \leq L - 1$$

or

$$II = \sum_{i=0}^{L-1+m} \left\{\cos \omega_0 m + \cos \omega_0 (2i - m + 2\Delta)\right\}, \qquad 0 > m > -L + 1.$$

Evaluating the finite summations I and II according to the identities in the appendix and substituting $\Delta = 1$:

$$I = (L - m) \cos \omega_0 m + \frac{\sin \omega_0 (L - m)}{\sin \omega_0} \cos [\omega_0 (L + 1)],$$

$$0 \leq m \leq L - 1$$

$$II = (L + m) \cos \omega_0 m + \frac{\sin \omega_0 (L + m)}{\sin \omega_0} \cos [\omega_0 (L + 1)],$$

$$0 > m > -L + 1.$$

By employing the absolute value notation $|m|$, the expressions I and II may be combined and the left-hand side of equation 42 written as

$$\left[(L - |m|) \cos \omega_0 m + \left[\frac{\cos \omega_0 (L + 1)}{\sin \omega_0}\right] \sin \omega_0 (L - |m|)\right],$$

$$0 \leq |m| \leq L - 1.$$

Substituting this expression into equation 41 gives the final value for ϕ_4:

$$\phi_4 = \frac{2a*^2 \nu^2}{L^2} \left[(L - |m|) \cos \omega_0 m + \left(\frac{\cos \omega_0 (L + 1)}{\sin \omega_0}\right) \sin \omega_0 (L - |m|)\right]$$

$$+ \mu c^2 \nu^2 L \, \delta(m), \qquad 0 \leq |m| \leq L - 1. \tag{43}$$

Now substituting equations 40 and 43 into equation 28, the autocorrelation function becomes

$$\phi_r(m) = E\left\{r(k|\theta)\right\} E\left\{r(k + m|\theta)\right\} + \frac{A^2 \mu c^2 L}{2} \cos \omega_0 m$$

$$-\frac{A^2 \mu c^2}{2} S_L(\omega_0) \cos[\omega_0(2k - m - L - 1) + 2\theta] + \mu c^2 \nu^2 L\, \delta(m)$$

$$+ \frac{2a^{*2} \nu^2}{L^2} \left\{(L - |m|) \cos \omega_0 m + \left(\frac{\cos \omega_0 (L+1)}{\sin \omega_0}\right) \sin \omega_0 (L - |m|)\right\},$$

$$|m| \leq L - 1. \qquad (44)$$

Thus, the covariance function as given by equation 22 becomes

$$\gamma_r(m) = \gamma_1(m) + \gamma_2(m) + \gamma_3(m) + \gamma_4(m) + \gamma_5(k, m),$$

$$0 \leq |m| \leq L - 1 \qquad (45)$$

where

$$\gamma_1(m) = \mu c^2 \nu^2 L\, \delta(m) \qquad (46a)$$

$$\gamma_2(m) = \left[\frac{A^2 \mu c^2 L}{2}\right] \cos \omega_0 m \qquad (46b)$$

$$\gamma_3(m) = \frac{2a^{*2} \nu^2}{L^2} (L - |m|) \cos \omega_0 m \qquad (46c)$$

$$\gamma_4(m) = \frac{2a^{*2} \nu^2}{L^2} \left(\frac{\cos \omega_0 (L+1)}{\sin \omega_0}\right) \sin \omega_0 (L - |m|) \qquad (46d)$$

$$\gamma_5(k, m) = -\left[\frac{A^2 \mu c^2}{2}\right] S_L(\omega_0) \cos[\omega_0(2k - m - L - 1) + 2\theta] \qquad (46e)$$

For lags, $|m| \geq L$, the covariance function becomes

$$\gamma_r(m) = \gamma_2(m) + \gamma_5(k, m) \qquad (47)$$

The origin of these terms is easily understood by considering the steady-state filter response as the superposition of two processes: (a) a fixed filter converged at the W_j^*, and (b) a broadband "noisy" filter due to the misadjustment noise. Thus, the γ_1 term is due to the uncorrelated noise sequence passing through the broadband noisy filter. The result is to produce an uncorrelated noise sequence in the filter output which is scaled by the product of total misadjustment noise power, $\mu c^2 L$, and white noise input power, ν^2. The term γ_2 is produced by the sinusoid (of power $A^2/2$) passing through the broadband filter. As such it represents a pure sinusoidal term in the output at ω_0 whose power is scaled by the sinusoid power and total misadjustment power. The term γ_3 is produced by the input white noise within the bandwidth of the converged filter. This produces a narrowband noise term in the filter output centered at the same frequency as the sinusoid, but with a bandwidth of the

converged filter. The finite bandwidth of this noise process is evidenced by the triangular weighting on the γ_3 component. The terms γ_4 and γ_5 result from the finite number of predictor weights and are produced, respectively, by uncorrelated noise passing through the converged filter and the sinusoid passing through the broadband filter. Note that $\gamma_1 - \gamma_4$ are not functions of time argument k and therefore represent stationary components of the filter output time series. However, γ_5 is a function of k and as such represents a nonstationary component of the output sequence.

The terms γ_1, γ_2, and γ_3 are contained in the covariance expression derived by Medaugh [1]. However, the function defined by $\gamma_r(m)$ in this paper contains the additional terms γ_4 and γ_5, which will next be examined in detail.

Origin of Nonstationary Covariance Terms

Consider Figure 4, which illustrates the case for an integral number of signal cycles contained within the filter length L. Note that for this case, $S_L(\omega_0) = 0$ and thus the $\gamma_5(k, m)$ vanishes. This corresponds to the continuous case that the integral over any number of full sinusoid cycles must vanish. As k increases, the signal progresses through the filter

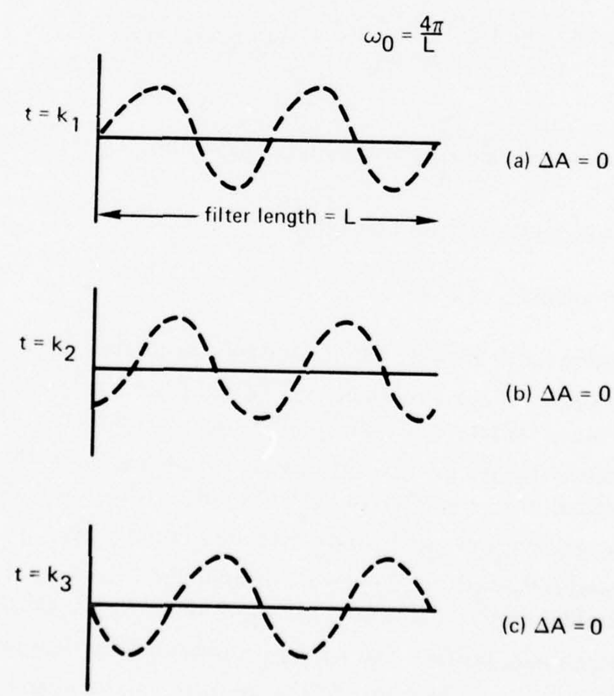

Figure 4. Progression of signal through filter for integer (n = 2) number of cycles.

but the integer characteristic of signal cycles remains constant. Thus $\gamma_5 = 0$, and the resulting expression for the covariance $\gamma_r(m)$ given by equation 45 is independent of absolute time index k. As ω_0 changes slightly, however, an incomplete cycle of signal data is held within the filter. (See Figure 5.) The quantity ΔA is the measure of excess sinusoidal cycles contained in the filter at the times $t = k_1, k_2, k_3$. For the case $\omega_0 = 4\pi/L$ in Figure 4, there is no cycle excess for any value of t. However, the absolute time value k causes an uncancelled cycle excess when the frequency $\omega_0 \neq 2n\pi/L$. Furthermore, the "polarity" of ΔA oscillates between its extrema with a frequency given by $2\omega_0$. This phenomenon can be seen by tracing the cycle excess in (a), (b) and (c) of Figure 5. From (a) to (c), the sinusoid itself has traversed $\pi/2$ radians through the filter, while the cycle excess has gone from its minimum to its maximum, a transition of π radians for a sinusoid. This is reflected in the time-dependent term $\gamma_5(k, m)$ which has oscillations at a frequency $2\omega_0$. For fixed lag m_0, the value of $\gamma_5(k, m_0)$ will oscillate about its mean value with frequency $2\omega_0$ as the time index k increases. For fixed time value k_0, the function $\gamma_5(k_0, m)$ behaves cosinusoidally as the lag m is increased.

Being dependent upon the time index k, $\gamma_5(k, m)$ as defined by equation 46, represents a nonstationary process occurring within the filter. This is due to the finite length of

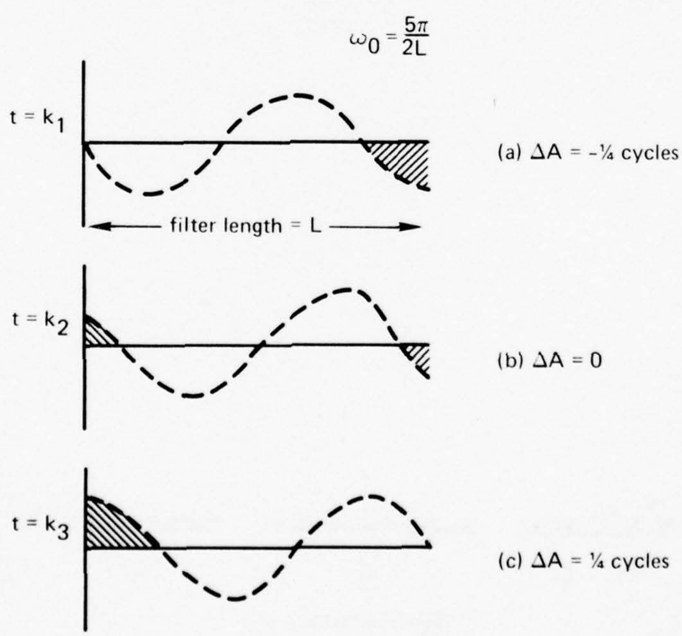

Figure 5. Progression of signal through filter for non-integer number of cycles.

the filter being "mismatched" to the signal frequency. However, there are two reasons why this term essentially can be neglected:

1. As time k increases, the average over k of $\gamma_5(k, m)$ is zero.

2. The amplitude of $\gamma_5(k,m)$ for all k, m is typically bounded much smaller than the amplitudes of the other terms in the covariance expression 45.

Figure 6 shows a plot of

$$\frac{S_L(\omega_0)}{L} = \frac{\sin \omega_0 L}{L \sin \omega_0}$$

vs. ω_0 which illustrates the frequency intervals for which $\gamma_5(k, m)$ may be neglected. For this figure L = 64, but the plot is representative of all values of L. The envelope of $S_L(\omega_0)/L$ decays quickly and is bounded by $1/L$ for ω_0 near $\pi/2$. For purposes of comparison, consider $\gamma_2(m)$ which is another narrowband component of the covariance expression equation 45. It is seen that the amplitude of $\gamma_5(k, m)$ is much smaller than $A^2\mu c^2 L/2$ for frequencies away from the DC bin by forming the ratio of the two amplitudes:

$$\frac{\text{Amp }[\gamma_5(k, m)]}{\text{Amp }[\gamma_2(m)]} = \frac{\frac{A^2 \mu c^2}{2} \cdot S_L(\omega_0)}{\frac{A^2 \mu c^2 L}{2}} = \frac{1}{L} S_L(\omega_0).$$

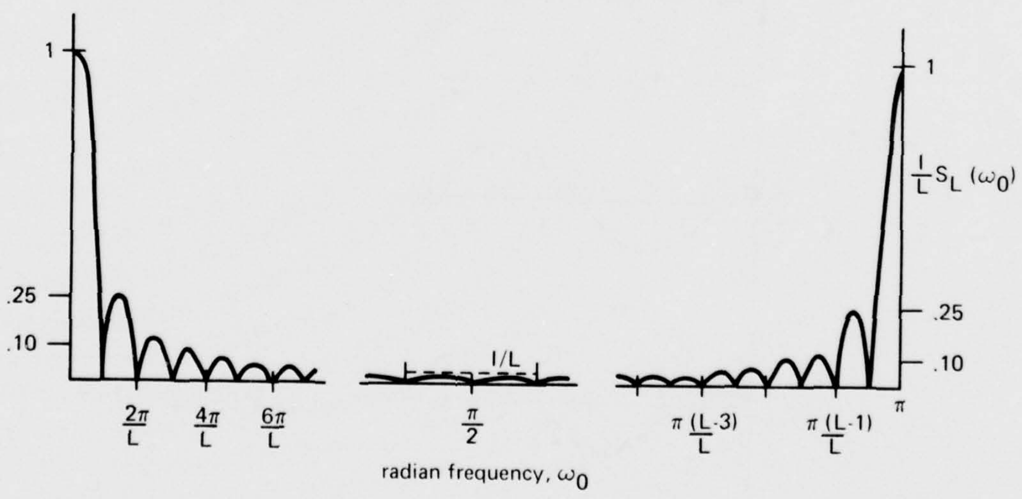

Figure 6. Plot of $S_L(\omega_0)/L$ vs. ω_0 for L = 64.

Thus, the amplitude ratio of $\gamma_1(k, m)$ to the narrowband component $\gamma_2(m)$ is given by the plot of Figure 6. For $\pi/L < \omega_0 < \pi(L-1)/L$, the $\gamma_5(k, m)$ term may be neglected without introducing appreciable error into the covariance function $\gamma_r(m)$.

The second term under consideration, $\gamma_4(m)$ as given by equation 47, is next to be examined. Its effects upon the total covariance function can best be seen by comparison with the narrowband covariance component $\gamma_3(m)$. This term and $\gamma_2(m)$ both contain the scale factor

$$\frac{2a^{*2}\nu^2}{L^2}$$

and thus may be compared easily. This is shown in Figures 7a, 7b and 7c which compares the two terms as a function of lag (m) for three different frequencies (a filter length L = 1000 was chosen). Figures 7a and 7b illustrate that the narrowband component $\gamma_3(m)$ dominates $\gamma_2(m)$ for $\omega_0 = \pi/10$ and $\omega_0 = \pi/100$. The narrowband component has a triangular envelope due to the $(L - |m|)$ weighting and thus only as the lag $|m|$ approaches L does $\gamma_4(m)$ become appreciable in magnitude to $\gamma_3(m)$. For $\omega_0 = \pi/1000$, however, Figure 7c shows that $\gamma_4(m)$ becomes appreciable in relative magnitude at about m = 0.2L. This frequency represents a line at the boundary between the DC bin and bin centered around $\omega = 2\pi/L = 2\pi/1000$. For this frequency, the $\gamma_4(m)$ should not be disregarded, especially for large m.

A similar behavior appears as ω_0 approaches π radians/sample interval. The term $\gamma_4(m)$ is now a narrow band component with frequency approaching π, but with an envelope similar to that of $\gamma_4(m)$ in Figure 7c. If ω_0 is located at the bin boundary between the $\omega = \pi$ and $\omega = (L - 2)\pi/L$ bins, then $\gamma_4(m)$ will be appreciable in magnitude for mid-range values of m and should not be disregarded.

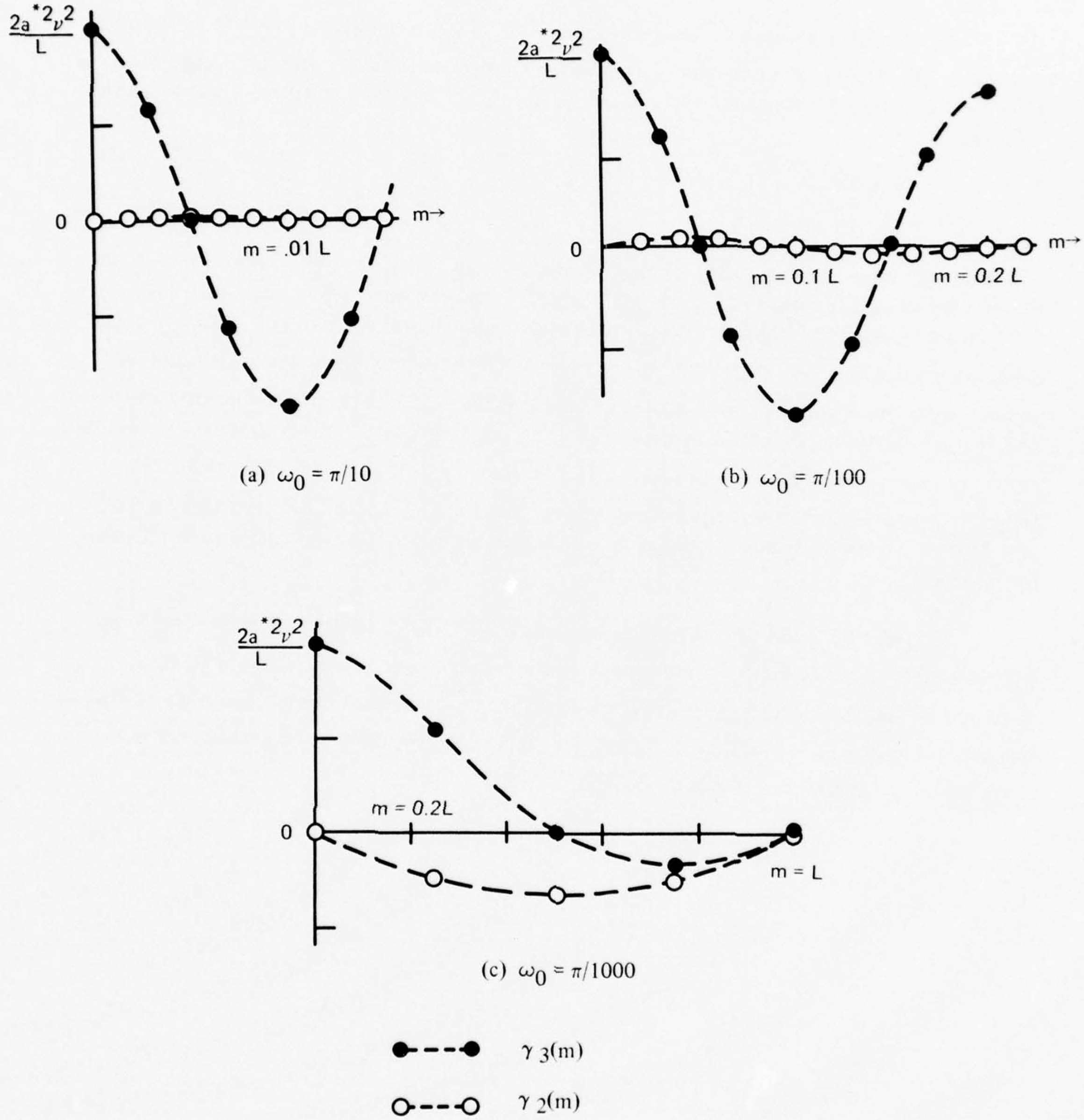

Figure 7. Relative values of covariance function components (a) $\omega_0 = 100\ \pi/L$, (b) $\omega_0 = 10\ \pi/L$, (c) $\omega_0 = \pi/L$ (L = 1000).

4. CALCULATION OF DFT MEAN AND VARIANCE FOR ALE OUTPUT

This section applies the covariance function toward obtaining expressions for the ALE filter output DFT mean and variance. The general outline of Section 2 is followed here, and the derivations from Section 3 are utilized.

As derived in Section 3, the following expression for the covariance sequence, $\gamma_r(m)$, will be used in the subsequent DFT mean and variance calculations:

$$\gamma_r(m) = \mu c^2 \nu^2 L\,\delta(m) + \left\{\frac{A^2 \mu c^2}{2} L + \frac{2a^{*2} \nu^2}{L^2}(L-|m|)\right\} \cos \omega_0 m$$

$$+ \frac{2a^{*2} \nu^2}{L^2}\left(\frac{\cos \omega_0 (L+1)}{\sin \omega_0}\right) \sin \omega_0 (L-|m|)\,, \quad 0 \leq |m| \leq L-1$$

$$= \frac{A^2 \mu c^2 L}{2} \cos \omega_0 m\,, \quad |m| \geq L\,. \tag{48}$$

In this section the DFT component means, denoted by $E(u)$ and $E(v)$, are derived, where u and v are defined by equations 15a and 15b, respectively. Additionally, the DFT component variances, σ_u^2 and σ_v^2, are derived.

DERIVATION OF COMPONENT MEANS

Begin by deriving the expectation of u, the real component of the output DFT at frequency ω_0, for the case of signal present. The notation

$$S_K(\omega_0) = \frac{\sin \omega_0 K}{\sin \omega_0}$$

will be used, as will the solution for W_j^*, the converged value of the j^{th} weight, given by

$$W_j^* = \frac{2a^*}{L} \cos [\omega_0 (j+\Delta)]\ ,$$

where the delay is Δ and the filter length is L.

Thus, for a DFT of K points,

$$E\{u\} = E\left\{\sum_{k=0}^{K-1} r(k|\theta) \cos \omega_0 k\right\} = E\left\{\sum_{k=0}^{K-1}\left[\sum_{j=0}^{K-1} W_j(k)\, x(k-j-A)\right] \times \cos \omega_0 k\right\}$$

$$= \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} E \; \{[W_j{}^* + V_j(k)] \; [A \sin [\omega_0 (k-j-\Delta) + \theta]$$

$$+ n(k-j-\Delta)]\} \cos \omega_0 k$$

$$= \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} \Big\{ A\, W_j{}^* \sin [\omega_0 (k-j-\Delta) + \theta] + A\, E \{V_j(k)\}$$

$$\times \sin [\omega_0 (k-j-\Delta) + \theta] + W_j{}^* E \{n(k-j-\Delta)\}$$

$$+ E \{V_j(k)\, n(k-j-\Delta)\}\Big\} \cos \omega_0 k$$

$$= A \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} W_j{}^* \sin [\omega_0 (k-j-\Delta) + \theta] \cos \omega_0 k$$

$$= \frac{2a^* A}{L} \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} \sin [\omega_0 (k-j-\Delta) + \theta] \cos [\omega_0 (j+\Delta)] \cos \omega_0 k$$

$$= \frac{a^* A}{L} \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} \sin [\omega_0 (k-j-\Delta) + \theta] \{\cos \omega_0 (j-k+\Delta)$$

$$+ \cos \omega_0 (j+k+\Delta)\}$$

$$= \frac{a^* A}{2L} \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} \Big\{ \sin [2\omega_0 (j-k+\Delta) + \theta] + \sin [\theta]$$

$$+ \sin [\omega_0 (2j + 2\Delta) - \theta] + \sin [\omega_0 (2k) + \theta] \Big\}$$

$$= \frac{a^* A}{2L} \sum_{k=0}^{K-1} \Big\{ S_K(\omega_0) \sin [\omega_0 (K-1) - 2\omega_0 (k-\Delta) + \theta] + K \sin \theta$$

$$+ S_K(\omega_0) \sin [\omega_0 (K-1) + 2\omega_0 \Delta - \theta] + K \sin [2\omega_0 k + \theta] \Big\}$$

$$= \frac{a^* A}{2L} \Big\{ K^2 \sin \theta - S_K{}^2 (\omega_0) \sin [\omega_0 (K-1) - \omega_0 (K-1)$$

$$- 2\omega_0 \Delta - \theta] + K\, S^2{}_K (\omega_0) \sin [\omega_0 (K-1) + 2\omega_0 \Delta - \theta]$$

$$+ K\, S_K (\omega_0) \sin [\omega_0 (K-1) + \theta]$$

Let $\Delta = 1$ for the white noise delay value. This gives the following:

$$E(u) = \frac{a^* AK^2}{2L} \sin\theta + \frac{a^* A}{2L} \Big\{ S_K{}^2(\omega_0) \sin(2\omega_0 + \theta)$$

$$+ K\, S_K(\omega_0) \sin[\omega_0(K+1) - \theta] + K\, S_K(\omega_0) \sin[\omega_0(K-1) + \theta] \Big\}$$

$$= \frac{a^* AK^2}{2L} \sin\theta + \frac{a^* A}{2L} S_K{}^2(\omega_0) \sin(2\omega_0 + \theta)$$

$$+ \frac{a^* AK}{L} S_K(\omega_0) \sin\omega_0 K \cos(\omega_0 - \theta) \ . \tag{49}$$

If ω_0 is bin-centered, $S_K(\omega_0) = 0$ and $E(u)$ reduces to

$$E(u) = \frac{a^* AK^2}{2L} \sin\theta \ . \tag{50}$$

Note that when the filter length and DFT length are equal, $K = L$, and

$$E(u) = \frac{a^* AL}{2} \sin\theta$$

which agrees with the expression derived in reference 1.

By the same analysis, $E(v)$ may be shown to be given by

$$E(v) = \frac{Aa^* K^2}{2L} \cos\theta - \frac{Aa^*}{2L} S_K{}^2(\omega_0) \cos[2\omega_0 - \theta]$$

$$+ \frac{a^* AK}{L} S_K(\omega_0) \sin\omega_0 K \sin(\omega_0 - \theta) \ . \tag{51}$$

If ω_0 is bin centered and $K = L$, then equation 51 reduces to

$$E(v) = \frac{a^* AL}{2} \cos\theta \tag{52}$$

which agrees with reference 1.

DERIVATION OF COMPONENT VARIANCES

Now, first derive an expression for the variance, $\sigma_u{}^2$, of the real component of the output DFT at ω_0 under conditions of signal present. The expression for $\sigma_u{}^2$ has been derived in section 2 and is given by

$$\sigma_u{}^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} \gamma_r(\ell - k) \cos\omega_0 k \cos\omega_0 \ell, \qquad K \leq L. \tag{53}$$

where K = number of points in the DFT and K is less than or equal to L, the length of the adaptive filter. All subsequent derivations assume $K \leq L$. Substituting equation 48 into equation 53 one obtains

$$\sigma_u^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} \Big\{ a\,\delta(\ell-k) + b \cos\omega_0(\ell-k) + c\,(L-|\ell-k|)\cos\omega_0(\ell-k)$$

$$+ cd \sin\omega_0\,(L-|\ell-k|)\Big\} \cos\omega_0 k \cos\omega_0 \ell \tag{54}$$

where

$$a = \mu c^2 \nu^2 L \ , \quad c = \frac{2a^{*2}\nu^2}{L^2}$$

$$b = \frac{\mu c^2 LA^2}{2} \ , \quad d = \frac{\cos\omega_0\,(L+1)}{\sin\omega_0} \tag{55}$$

The variance expression in equation 54 can be rewritten as

$$\sigma_u^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 \tag{56}$$

where

$$\sigma_1^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} a\,\delta(\ell-k)\cos\omega_0 k \cos\omega_0 \ell \tag{57a}$$

$$\sigma_2^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} b\cos\omega_0(\ell-k)\cos\omega_0 k \cos\omega_0 \ell \tag{57b}$$

$$\sigma_3^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} c\,(L-|\ell-k|)\cos\omega_0(\ell-k)\cos\omega_0 k \cos\omega_0 \ell \tag{57c}$$

$$\sigma_4^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} cd\sin\omega_0\,(L-|\ell-k|)\cos\omega_0 k \cos\omega_0 \ell\ . \tag{57d}$$

The evaluation of these terms is quite complicated and thus each term will be treated in a separate subsection.

A. Evaluation of $\sigma_1{}^2$

Begin with $\sigma_1{}^2$ as given by equation 57a:

$$\sigma_1{}^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} a\,\delta(\ell-k) \cos \omega_0 k \cos \omega_0 \ell \tag{58}$$

$$= a \sum_{k=0}^{K-1} \cos^2 \omega_0 k = \frac{a}{2} \sum_{k=0}^{K-1} [1 + \cos 2\omega_0 k]$$

$$\sigma_1{}^2 = \frac{aK}{2} + \frac{a}{2} S_K(\omega_0) \cos \omega_0 (K-1) \ . \tag{59}$$

B. Evaluation of $\sigma_2{}^2$

Begin with $\sigma_2{}^2$ as given by equation 57b:

$$\sigma_2{}^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} b \cos \omega_0(\ell-k) \cos \omega_0 k \cos \omega_0 \ell \tag{60}$$

$$= \sum_k \sum_\ell b\,[\cos \omega_0 \ell \cos \omega_0 k + \sin \omega_0 \ell \sin \omega_0 k] \cos \omega_0 k$$

$$\times \cos \omega_0 \ell$$

$$= \sum_k \sum_\ell b \left[\cos^2 \omega_0 \ell \cos^2 \omega_0 k + \frac{1}{4} \sin 2\omega_0 k \sin 2\omega_0 \ell\right]$$

$$= \frac{b}{4} \sum_k \sum_\ell [1 + \cos 2\omega_0 \ell + \cos 2\omega_0 k + \cos 2\omega_0(\ell-k)]$$

$$= \frac{b}{4} \sum_{k=0}^{K-1} [K + S_K(\omega_0) \cos \omega_0(K-1) + K \cos 2\omega_0 k$$

$$+ S_K(\omega_0) \cos [\omega_0(K-1) - 2\omega_0 k]]$$

$$= \frac{b}{4} \left[K^2 + K\,S_K(\omega_0) \cos \omega_0(K-1) + K\,S_K(\omega_0) \cos \omega_0(K-1)\right.$$

$$\left. + S^2{}_K(\omega_0)\right]$$

$$= \frac{b K^2}{4} + \frac{K b S_K(\omega_0)}{2} \left[\cos \omega_0(K-1) + \frac{S_K(\omega_0)}{2K}\right] . \tag{61}$$

C. Evaluation of $\sigma_3{}^2$

Start with $\sigma_3{}^2$ as given in equation 57c:

$$\sigma_3{}^2 = \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} c\,(L - |\ell - k|) \cos \omega_0(\ell - k) \cos \omega_0 k \cos \omega_0 \ell \ . \tag{62}$$

Using a series of trigonometric identities, this can be written as

$$\sigma_3{}^2 = \frac{cL}{4} \sum_{\ell=0}^{K-1} \sum_{k=0}^{K-1} [1 + \cos 2\omega_0(\ell - k) + \cos 2\omega_0 k + \cos 2\omega_0 \ell]$$

$$- \frac{c}{4} \sum_{\ell=0}^{K-1} \sum_{k=0}^{K-1} |\ell - k| \, [1 + \cos 2\omega_0(\ell - k) + \cos 2\omega_0 k$$

$$+ \cos 2\omega_0 \ell] \tag{63}$$

$$= S_1 - S_2 \ . \tag{63a}$$

Sum the first summation S_1 over the k index:

$$S_1 = \frac{cL}{4} \sum_{\ell=0}^{K-1} [K + S_K(\omega_0) \cos [\omega_0(K-1) - 2\omega_0 \ell]$$

$$+ S_K(\omega_0) \cos \omega_0(K-1) + K \cos 2\omega_0 \ell] \ .$$

Then sum over ℓ:

$$S_1 = \frac{cL}{4} \left[K^2 + S_K{}^2(\omega_0) + 2K\, S_K(\omega_0) \cos \omega_0(K-1)\right] . \tag{64}$$

Now expand the second summation S_2 from equation 63 and remove the absolute value notation:

$$S_2 = \frac{c}{4} \sum_{\ell=0}^{K-1} \sum_{k=0}^{\ell-1} (\ell-k) \left[1 + \cos 2\omega_0(\ell-k) + \cos 2\omega_0 k + \cos 2\omega_0 \ell\right]$$

$$+ \frac{c}{4} \sum_{\ell=0}^{K-1} \sum_{k=\ell}^{K-1} (k-\ell) \left[1 + \cos 2\omega_0(\ell-k) + \cos 2\omega_0 k\right.$$

$$\left. + \cos 2\omega_0 \ell\right] \qquad (65)$$

$$= S_{2,1} + S_{2,2} \; .$$

Expand $S_{2,1}$ by summing over k:

$$S_{2,1} = \frac{c}{4} \sum_{\ell=0}^{K-1} \left[\ell^2 + \ell S_K(\omega_0) \cos\left[\omega_0(K-1) - 2\omega_0 \ell\right]\right.$$

$$+ \ell S_K(\omega_0) \cos \omega_0(K-1) + \ell^2 \cos 2\omega_0 \ell - \sum_{k=0}^{\ell-1} k$$

$$- \sum_{k=0}^{\ell-1} k \cos 2\omega_0(k-\ell) - \sum_{k=0}^{\ell-1} k \cos 2\omega_0 k$$

$$\left. - \cos 2\omega_0 \ell \sum_{k=0}^{\ell-1} k \right] .$$

The final four summations may be written in closed form with the aid of the identities in the appendix, giving:

$$S_{2,1} = \frac{c}{4} \sum_{\ell=0}^{K-1} \left[\ell^2 + S_K(\omega_0)\, \ell \cos\left[\omega_0(K-1) - 2\omega_0 \ell\right]\right.$$

$$+ S_K(\omega_0)\, \ell \cos \omega_0(K-1) + \ell^2 \cos 2\omega_0 \ell - \frac{\ell(\ell-1)}{2}$$

$$+ \frac{1}{2} \frac{d}{d\omega_0} \left\{ \frac{\sin \omega_0 \ell}{\sin \omega_0} \sin \omega_0(\ell+1) \right\}$$

$$\left. - \frac{1}{2} \frac{d}{d\omega_0} \left\{ \frac{\sin \omega_0 \ell}{\sin \omega_0} \sin \omega_0(\ell-1) \right\} - \frac{\ell(\ell-1)}{2} \cos 2\omega_0 \ell \right] .$$

Combining similar terms and grouping the derivative terms together:

$$\frac{1}{2}\frac{d}{d\omega_0}\left\{\frac{\sin\omega_0\ell}{\sin\omega_0}[\sin\omega_0(\ell+1)-\sin\omega_0(\ell-1)]\right\}$$

$$=\frac{1}{2}\frac{d}{d\omega_0}\left\{\frac{\sin\omega_0\ell}{\sin\omega_0}[2\sin\omega_0\cos\omega_0\ell]\right\}$$

$$=\frac{d}{d\omega_0}\left\{\sin\omega_0\ell\cos\omega_0\ell\right\}=\ell\cos 2\omega_0\ell\ ,$$

from which we get:

$$S_{2,1}=\frac{c}{4}\sum_{\ell=0}^{K-1}\left\{\frac{\ell^2+\ell}{2}+\frac{\ell^2+3\ell}{2}\cos 2\omega_0\ell\right.$$
$$+S_K(\omega_0)\,[\ell\cos[2\omega_0\ell-\omega_0(K-1)]$$
$$\left.+\ell\cos\omega_0(K-1)]\right\}\ . \qquad (66)$$

Now sum over ℓ:

$$S_{2,1}=\frac{c}{4}\left[\frac{K^3-K}{6}+\frac{1}{2}\left\{-\frac{1}{4}\frac{d^2}{d\omega_0{}^2}S_K(\omega_0)\cos\omega_0(K-1)\right\}\right.$$
$$+\frac{3}{2}\left\{\frac{1}{2}\frac{d}{d\omega_0}S_K(\omega_0)\sin\omega_0(K-1)\right\}$$
$$\left.+K(K-1)\frac{S_K(\omega_0)}{2}\cos\omega_0(K-1)\right]$$

giving

$$S_{2,1}=\frac{c}{4}\left[\frac{K^3-K}{6}+\frac{3K^2-K-2}{4}S_K(\omega_0)\cos\omega_0(K-1)\right.$$
$$+\frac{K+2}{4}S_K'(\omega_0)\sin\omega_0(K-1)$$
$$\left.-\frac{1}{8}S_K''(\omega_0)\cos\omega_0(K-1)\right]\ . \qquad (67)$$

Now return to $S_{2,2}$ given by equation 65 and make the substitution $n=k-\ell$, giving

$$S_{2,2}=\frac{c}{4}\sum_{\ell=0}^{K-1}\sum_{n=0}^{K-1-\ell}[n+n\cos 2\omega_0 n+n\cos 2\omega_0(n+\ell)+n\cos 2\omega_0\ell] \qquad (68)$$

Summing over n:

$$S_{2,2} = \frac{c}{4}\sum_{\ell=0}^{K-1}\left[\frac{(K-\ell)(K-\ell-1)}{2} + \frac{1}{2}\frac{d}{d\omega_0}\left\{\frac{\sin\omega_0(K-\ell)}{\sin\omega_0}\sin\omega_0(K-1-\ell)\right\}\right.$$

$$+\frac{1}{2}\frac{d}{d\omega_0}\left\{\frac{\sin\omega_0(K-\ell)}{\sin\omega_0}\sin\omega_0(K-1+\ell)\right\}$$

$$\left.+\frac{(K-\ell)(K-1-\ell)}{2}\cos 2\omega_0\ell\right]$$

$$S_{2,2} = \frac{c}{4}\sum_{\ell=0}^{K-1}\left\{\frac{(K^2-K)+\ell(1-2K)+\ell^2}{2}\right.$$

$$+\frac{d}{d\omega_0}\left[\frac{\sin\omega_0(K-\ell)\sin\omega_0(K-1)\cos\omega_0\ell}{\sin\omega_0}\right]$$

$$\left.+\frac{(K^2-K)+\ell(1-2K)+\ell^2}{2}\cos 2\omega_0\ell\right\} .$$

Now sum over ℓ:

$$S_{2,2} = \frac{c}{4}\left\{\left[\frac{K^3-K^2}{2} + \frac{K(1-2K)(K-1)}{4} + \frac{2K^3-3K^2+K}{12}\right]\right.$$

$$+\left[\sum_{\ell=0}^{K-1}\frac{d}{d\omega_0}\left\{\frac{\sin\omega_0(K-1)\sin\omega_0(K-\ell)\cos\omega_0\ell}{\sin\omega_0}\right\}\right]$$

$$+\left[\left(\frac{K^2-1}{8}\right)S_K(\omega_0)\cos\omega_0(K-1)\right.$$

$$-\left(\frac{K-1}{2}\right)S_K'(\omega_0)\sin\omega_0(K-1)$$

$$\left.\left.-\frac{1}{8}S_K''(\omega_0)\cos\omega_0(K-1)\right]\right\} . \qquad (69)$$

The summation of the derivative term leads to a complicated expression, to say the least. To evaluate equation 69, the differentiation of the second term was carried out, then the resulting terms summed over the ℓ index. The $S_K'(\omega_0)$ and $S_K''(\omega_0)$ terms appearing were expanded using equations A10 and A15 from the Appendix. Then terms were grouped according to frequency dependence, giving

$$S_{2,2} = \frac{c}{4}\left[\frac{K^3-K}{6}\right]$$

$$+\frac{c}{4} S_K(\omega_0) \left\{ \left[\frac{6K^2+2K-3}{8}\right] \cos\omega_0(K-1) \right.$$

$$+ \left(\frac{K+1}{2}\right) S_K(\omega_0) + \frac{\sin\omega_0(K-2)}{4\sin\omega_0}$$

$$+ \left[\frac{(7K-3)\cos\omega_0 - (K+1)}{8\sin\omega_0}\right] \sin\omega_0(K-1)$$

$$+ \left[\cos\omega_0 + \frac{K^2-K}{4}\right] \cos\omega_0 K$$

$$\left. - \left(\frac{\cos\omega_0}{4\sin^2\omega_0}\right) \cos\omega_0(K-2) \right\}$$

$$+\frac{c}{4}\left\{ \frac{K}{4\sin^2\omega_0} \left[1 - \cos^2\omega_0 + \cos\omega_0 \cos\omega_0(2K-1)\right] \right.$$

$$+ \frac{K\cos\omega_0 K}{4\sin\omega_0} \left[\frac{(1+\cos\omega_0)\cos\omega_0(K-1)}{2\sin\omega_0}\right.$$

$$\left.\left. + (K-1)\cos\omega_0 K\right]\right\} . \qquad (70)$$

The expression for $\sigma_3{}^2$ is now a combination of equations 64, 67 and 70. Rewrite equation 64 as

$$S_1 = \frac{cLK^2}{4} + \frac{c}{4} S_K(\omega_0)\, F_1(KL, \omega_0) . \qquad (71)$$

First observe that if

$$\omega_0 = \frac{\pi m}{K} \ (m = \text{integer}),$$

then

$$S_K(\omega_0) = \frac{\sin\omega_0 K}{\sin\omega_0} = 0 .$$

This is the case of signal frequency being exactly bin-centered or exactly on bin boundaries of the K-point FFT. When $S_K(\omega_0)$ vanishes then S_1 reduces to

$$S_1 = \frac{cLK^2}{4} . \qquad (72)$$

Even when ω_0 is not at these critical frequencies, the function $F_1(KL,\omega_0)$ is much less than LK^2 by a factor of $1/K$. The notation KL in $F_1(KL,\omega_0)$ represents the highest order products appearing in the function. As $\omega_0 \rightarrow 0$ or $\omega_0 \rightarrow \pi$, then $S_K(\omega_0) \rightarrow K$ and a term approaching a magnitude LK^2 does appear. However, as long as $\pi/L < \omega_0 < \pi(L-1)/L$ work can be done accurately with the approximation

$$S_K(\omega_0) \ll K$$

from which

$$S_K(\omega_0)\, F_1(KL,\omega_0) \ll K^2 L$$

which will allow the approximation to S_1 as given by equation 72.

The term $S_{2,1}$ from equation 67 may be rewritten as

$$S_{2,1} = \frac{c}{4}\left[\frac{K^3}{6}\right] - \frac{c}{4}\left[\frac{K}{6}\right] + S_K(\omega_0)\, F_2\left(K^2, K, \omega_0\right)$$
$$+ S_K'(\omega_0)\, F_3(K,\omega_0) + S_K''(\omega_0)\, F_4(\omega_0)$$

The behavior of $S_K'(\omega_0)$ and $S_K''(\omega_0)$ is examined in the Appendix. The result is that as long as ω_0 is sufficiently removed from 0 or π, the expression for $S_{2,1}$ is dominated by the first term. Thus $S_{2,1}$ may be approximated by

$$S_{2,1} = \frac{cK^3}{24} \quad . \tag{73}$$

The same approach may be taken toward $S_{2,2}$ as given by equation 70, which may be rewritten as

$$S_{2,2} = \frac{c}{4}\left(\frac{K^3}{6}\right) - \frac{c}{4}\left[\frac{K^3}{6}\right] + \frac{c}{4} S_K(\omega_0)\, F_5(\omega_0, K^2, K)$$
$$+ \frac{c}{4} F_6(\omega_0, K^2, K) \quad . \tag{74}$$

For ω_0 not near the critical frequencies, the approximation becomes:

$$S_{2,2} \cong \frac{c}{4}\left[\frac{K^3}{6}\right] = \frac{cK^3}{24} \quad . \tag{75}$$

By using equations 63a, 72, 73 and 75 the useful (and workable) approximation to σ_3^2 then becomes:

$$\sigma_3^2 = S_1 - S_{2,1} - S_{2,2}$$

$$\cong \frac{cLK^2}{4} - \frac{cK^3}{12} = cK^2\left(\frac{L}{4} - \frac{K}{12}\right) . \qquad (76)$$

Note that when the transform length K equals L, the filter length, then

$$\sigma_3^2 = \frac{cL^3}{6}$$

which agrees with the result in Medaugh's derivation.

D. Evaluation of σ_4^2

The evaluation of σ_4^2 as given by equation 57d produces a result of the following form:

$$\sigma_4^2 = \frac{c}{2}[S_K(\omega_0)\,G_1(\omega_0,KL) + S_L(\omega_0)\,G_2(\omega_0,KL) + G_3(\omega_0,K,L)] \ .$$

Thus, for $\pi/L < \omega_0 < \pi(L-1)/L$ this term has a magnitude on the order $cKL/2$, which is much smaller than the expression for σ_3^2 as given by equation 76. Thus the working approximation to σ_4^2 will be

$$\sigma_4^2 \cong 0 \ . \qquad (77)$$

USEFUL VARIANCE EXPRESSION

The useful approximation to the variance of u is now obtainable. By the preceding arguments, the following approximations are valid for $\pi/L < \omega_0 < \pi(L-1)/L$:

By equation 59, $\sigma_1^2 \cong \frac{aK}{2} = \frac{\mu c^2 \nu^2 LK}{2}$. (78)

By equation 61, $\sigma_2^2 \cong \frac{bK^2}{2} = \frac{\mu c^2 A^2 LK^2}{8}$. (79)

By equation 76, $\sigma_3^2 \cong cK^2\left(\frac{L}{4} - \frac{K}{12}\right) = 2a^{*2}\nu^2\frac{K^2}{L^2}\left(\frac{L}{4} - \frac{K}{12}\right)$. (80)

By equation 77, $\sigma_4^2 \cong 0$. (81)

The expression for the approximation to σ_u^2 is

$$\sigma_u^2 \cong \frac{\mu c^2 \nu^2 LK}{2} + \frac{\mu c^2 A^2 LK^2}{8} + \frac{2a^{*2} \nu^2 K^2}{L^2} \left(\frac{L}{4} - \frac{K}{12}\right), K \leqslant L. \tag{82}$$

By the symmetry of the DFT operation, the extension to the variance of the imaginary component σ_v^2 may be made, giving

$$\sigma_v^2 = \sigma_u^2 \tag{83}$$

When the transform length, K, equals the filter length L, equation 82 becomes the variance expression derived in reference 1:

$$\sigma_u^2 = \frac{\mu c^2 \nu^2 L^2}{2} + \frac{\mu c^2 A^2 L^3}{8} + \frac{a^* \nu^2 L}{3} \tag{84}$$

where $\alpha = 2\mu c^2 L$ in reference 1.

Having obtained the results in this section one may proceed with developing the probability density function for the ALE output, as outlined in Section 2. The desired mean values $\bar{u}$ and $\bar{v}$ are given by equations 49 and 51, respectively, and useful approximations to the variances σ_u^2 and σ_v^2 are given by equations 82 and 83, respectively.

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APPENDIX

USEFUL CLOSED FORM EXPRESSIONS FOR FINITE SUMMATIONS

In deriving the variance and mean solutions, it is necessary to employ closed form expressions for finite summations of trigonometric terms. From reference 3, the formula obtained is:

$$\sum_{k=0}^{n} \sin(ka+b) = \frac{\sin\left(\frac{n+1}{2}a\right)\sin\left(\frac{n}{2}a+b\right)}{\sin\frac{a}{2}} \tag{A1}$$

$$\sum_{k=0}^{n} \cos(ka+b) = \frac{\sin\left(\frac{n+1}{2}a\right)\cos\left(\frac{n}{2}a+b\right)}{\sin\frac{a}{2}}. \tag{A2}$$

Specifically, let $n = K - 1$, $a = 2\omega_0$:

$$\sum_{k=0}^{K-1} \sin(2\omega_0 k + b) = \frac{\sin \omega_0 K}{\sin \omega_0} \sin[\omega_0 (K-1) + b]$$

$$= S_K(\omega_0) \sin[\omega_0 (K-1) + b] \ . \tag{A3}$$

$$\sum_{k=0}^{K-1} \cos(2\omega_0 k + b) = \frac{\sin \omega_0 K}{\sin \omega_0} \cos[\omega_0 (K-1) + b]$$

$$= S_K(\omega_0) \cos[\omega_0 (K-1) + b], \tag{A4}$$

where

$$S_K(\omega) = \frac{\sin \omega K}{\sin \omega} \ .$$

By taking derivatives of equations A3 and A4, the following relations may be obtained:

$$\sum_{k=0}^{K-1} k \cos(2\omega_0 k+b) = \frac{1}{2}\frac{d}{d\omega_0}\left\{S_K(\omega_0)\sin[\omega_0(K-1)+b]\right\} \tag{A5}$$

$$\sum_{k=0}^{K-1} k \sin(2\omega_0 k+b) = -\frac{1}{2}\frac{d}{d\omega_0}\left\{S_K(\omega_0)\cos[\omega_0(K-1)+b]\right\} \ . \tag{A6}$$

Further differentiation of equations A5 and A6 gives:

$$\sum_{k=0}^{K-1} k^2 \cos(2\omega_0 k + b) = -\frac{1}{4}\frac{d^2}{d\omega_0}\left\{S_K(\omega_0)\cos[\omega_0(K-1)+b]\right\} \qquad (A7)$$

$$\sum_{k=0}^{K-1} k^2 \sin(2\omega_0 k + b) = -\frac{1}{4}\frac{d^2}{d\omega_0^{\,2}}\left\{S_K(\omega_0)\sin[\omega_0(K-1)+b]\right\} \;. \qquad (A8)$$

Carrying out the differentiation in equation A5.

$$\sum_{k=0}^{K-1} k\cos(2\omega_0 k + b) = \frac{1}{2}\left\{S_K'(\omega_0)\sin[\omega_0(K-1)+b]\right.$$

$$\left. + (K-1)\,S_K(\omega_0)\cos[\omega_0(K-1)+b]\right\}$$

$$= \frac{K}{2}S_K(\omega_0)\cos[\omega_0(K-1)+b] + \frac{S_K'(\omega_0)}{2}\sin[\omega_0(K-1)+b]$$

$$- \frac{S_K(\omega_0)}{2}\cos[\omega_0(K-1)+b] \qquad (A9)$$

where

$$S_K'(\omega_0) = \frac{d}{d\omega_0}\left[\frac{\sin\omega_0 K}{\sin\omega_0}\right] = K\,\frac{\cos\omega_0 K}{\sin\omega_0} - S_K(\omega_0)\frac{\cos\omega_0}{\sin\omega_0} \qquad (A10)$$

The term $S_K(\omega_0)$ is $\ll K$ as long as $\pi/L < \omega_0 < \pi(L-1)/L$. The behavior of $S_K'(\omega_0)$ at these critical frequencies may be seen either from examining Figure 3 or taking the limit of both sides of equation A10 as $\omega_0 \to 0$ (or $\omega_0 \to \pi$):

$$\lim_{\omega_0 \to 0} S_K'(\omega_0) = \lim_{\omega_0 \to 0}\left[K\,\frac{\cos\omega_0 K}{\sin\omega_0} - S_K(\omega_0)\frac{\cos\omega_0}{\sin\omega_0}\right] .$$

Using L'Hopital's rule, the limit on the right is shown to be zero. Thus,

$$\lim_{\omega_0 \to 0} S_K'(\omega_0) = 0 \;. \qquad (A11)$$

Returning to equation A9, the limit of both sides is taken as $\omega \to 0$ and the result from equation A11 is used:

$$\lim_{\omega_0 \to 0} \sum_{k=0}^{K-1} k \cos(2\omega_0 k + b) = \lim_{\omega_0 \to 0} \left[\frac{K}{2} S_K(\omega_0) - \frac{S_K(\omega_0)}{2}\right] \cos\{\omega_0 (K-1) + b\}$$

giving

$$\sum_{k=0}^{K-1} k \cos(b) = \left[\frac{K^2}{2} - \frac{K}{2}\right] \cos(b) \cdot$$

For phase angle b = 0 this gives the familiar formula (reference 3)

$$\sum_{k=0}^{K-1} k \cos(0) = \sum_{k=1}^{K-1} k = \frac{K(K-1)}{2} . \qquad \text{(A12)}$$

Thus the maximum value of the summation in equation A5 is given by equation A12. For $\omega_0 = 0$ or $\omega = \pi$, the summation may introduce terms on the order of K^2 (K = filter length or number of points in DFT), dependent on the phase angle b. A similar analysis will show that the summation given by equation A6 has the same maximum value of $\frac{K(K-1)}{2}$ and this is obtained when $\omega_0 = 0, \pi$ and $b = \pi/2$. The value of both summations in equations A5 and A6 decays quickly for ω_0 away from 0 or π due to the weighting by the $S_K(\omega_0)$ terms.

Next are derived the bounds for the summations of equations A7 and A8. Carrying out the differentiation specified in equation A7,

$$\sum_{k=0}^{K-1} k^2 \cos(2\omega_0 k + b) = -\frac{1}{4} \frac{d^2}{d\omega_0^2} \{S_K(\omega_0) \cos[\omega_0 (k-1) + b]\}$$

$$= -\frac{1}{4} \frac{d}{d\omega} \{S_K'(\omega_0) \cos[\omega_0 (K-1) + b] - (K-1) S_K(\omega_0) \sin[\omega_0 (K-1) + b]\}$$

or

$$\sum_{k=0}^{K-1} k^2 \cos(2\omega_0 k + b) = \frac{K^2}{4} S_K(\omega_0) \cos[\omega_0 (K-1) + b]$$

$$+\frac{K}{2}\{S_K'(\omega_0) \sin[\omega_0 (K-1) + b]$$

$$- S_K(\omega_0) \cos[\omega_0 (K-1) + b]\} + \frac{1}{4}\{[S_K(\omega_0)$$

$$- S_K''(\omega_0)] \cos[\omega_0 (K-1) + b]$$

$$- 2 S_K'(\omega_0) \sin[\omega_0 (K-1) + b]\} \quad \text{(A13)}$$

The maximum for equation A13 occurs when $\omega_0 = 0$ and $b = 0$. Taking the limit of both sides of equation A13 and employing previously derived results gives

$$\sum_{k=0}^{K-1} k^2 = \frac{K^2}{4} \lim_{\omega_0 \to 0} S_K(\omega_0) - \frac{K}{2} \lim_{\omega_0 \to 0} S_K(\omega_0) + \frac{1}{4} \lim_{\omega_0 \to 0} [S_K(\omega_0) - S_K''(\omega_0)]$$

$$= \frac{K^3}{4} - \frac{K^2}{2} + \frac{1}{4}\left[K - \lim_{\omega_0 \to 0} S_K''(\omega_0)\right] . \quad \text{(A14)}$$

Thus, to find the bound the behavior of $S_K(\omega_0)$ in the vicinity of $\omega_0 \to 0$ must be examined. From equation A10,

$$S_K''(\omega_0) = \frac{d^2}{d\omega_0^2}\left[\frac{\sin \omega_0 K}{\sin \omega_0}\right] = \frac{d}{d\omega_0}\left[K \frac{\cos \omega_0 K}{\sin \omega_0} - S_K(\omega_0) \frac{\cos \omega_0}{\sin \omega_0}\right]$$

$$S_K''(\omega_0) = -K^2 S_K(\omega_0) + S_K(\omega_0)\left[\frac{1 + \cos^2 \omega_0}{\sin^2 \omega_0}\right]$$

$$- 2K\left[\frac{\cos \omega_0 K \cos \omega_0}{\sin^2 \omega_0}\right] . \quad \text{(A15)}$$

Taking the limit of both sides of equation A15:

$$\lim_{\omega_0 \to 0} S_K''(\omega_0) = -K^2 \lim_{\omega_0 \to 0} S_K(\omega_0)$$

$$+ \lim_{\omega_0 \to 0}\left\{\frac{S_K(\omega_0)[1 + \cos^2 \omega_0]}{\sin^2 \omega_0} - \frac{2K \cos \omega_0 K \cos \omega_0}{\sin^2 \omega_0}\right\}.$$

Application of L'Hopital's Rule (and steadfast perseverance) gives

$$\lim_{\omega_0 \to 0} S_K''(\omega_0) = -K^3 + \left(\frac{2K^3 + K}{3}\right)$$

$$= -\frac{K^3 + K}{3} \quad . \tag{A16}$$

Substituting this result in equation A14 gives

$$\sum_{k=0}^{K-1} k^2 = \frac{K^3}{4} - \frac{K^2}{2} + \frac{1}{4}\left[K + \frac{K^3 - K}{3}\right] = \frac{2K^3 - 3K^2 + K}{6} \quad . \tag{A17}$$

This last result may be factored, giving the familiar identity (reference 3)

$$\sum_{k=0}^{K-1} k^2 = \frac{(K-1)K(2K-1)}{6} \quad . \tag{A18}$$

The result from equation A17 shows that for $\omega_0 \to (0, \pi)$ summation equation A7 may introduce terms on the order of K^3. A similar analysis for the summation in equation A8 gives the same maximum value as $\omega_0 \to (0, \pi)$. As in the previous derivation, the summations equations A7 and A8 decay quickly as ω_0 is moved away from $\omega_0 = (0, \pi)$.